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AMENABLE GROUPS, TOPOLOGICAL ENTROPY AND BETTI NUMBERS

BY

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ABSTRACT

We investigate an analogue of the L^2-Betti numbers for amenable linear subshifts. The role of the von Neumann dimension shall be played by the topological entropy.

1. Introduction

Let Γ be a finitely generated group. Then the Hilbert space $l^2(\Gamma)$ has a natural left Γ-action by translations:

$$L_\gamma(f)(\delta) = f(\gamma^{-1}\delta).$$

Using the so-called von Neumann dimension we can assign a real number to any Γ-invariant linear subspace of $[l^2(\Gamma)]^n, n \in \mathbf{N}$ satisfying the following basic axioms [16].

1. **Positivity**: If $V \subset [l^2(\Gamma)]^n$ Γ-invariant linear subspace, then $\dim_\Gamma(V) \geq 0$. Also, $\dim_\Gamma(V) = 0$ if and only if $V = 0$.
2. **Invariance**: If $V \subset [l^2(\Gamma)]^n$, $W \subset [l^2(\Gamma)]^m$ and T is a Γ-equivariant isomorphism from V to a dense subset of W, then $\dim_\Gamma(V) = \dim_\Gamma(W)$.
3. **Additivity**: If Z is the orthogonal direct sum of V and W, then $\dim_\Gamma(Z) = \dim_\Gamma(V) + \dim_\Gamma(W)$.

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4. **Continuity**: If $V_1 \supset V_2 \supset \cdots$ is a decreasing sequence of Γ-invariant linear subspaces, then
$$\dim_\Gamma(\bigcap_{j=1}^{\infty} V_j) = \lim_{j\to\infty} \dim_\Gamma(V_j).$$
5. **Normalization**: $\dim_\Gamma[l^2(\Gamma)] = 1$.

There is an important application of the von Neumann dimension in algebraic topology due to Atiyah [1] (see also [4]). He defined certain invariants of finite simplicial complexes: the L^2-Betti numbers. The idea is the following. Let $\widetilde{K}$ be an infinite, simplicial complex with a free and simplicial Γ-action as covering transformations such that $\widetilde{K}/\Gamma = K$ is finite. Denote by $C^p_{(2)}(\widetilde{K})$ the Hilbert space of square-summable, real p-cochains of $\widetilde{K}$. Then one has the following differential complex of Hilbert spaces,
$$C^0_{(2)}(\widetilde{K}) \xrightarrow{d_0} C^1_{(2)}(\widetilde{K}) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n_{(2)}(\widetilde{K}),$$
where the d_p's are the usual coboundary operators.

Note that $C^p_{(2)}(\widetilde{K}) \cong [l^2(\Gamma)]^{|K_p|}$, where K_p denotes the set of p-simplices in K. Atiyah's L^2-Betti numbers are defined as
$$L_{(2)}b^p(K) = \dim_\Gamma \operatorname{Ker} d_p - \dim_\Gamma \operatorname{Im} d_{p-1}.$$
Let us list some basic results on the L^2-Betti numbers.

- (Dodziuk, [4]) If $\widetilde{K}$ and $\widetilde{L}$ are homotopic by a Γ-invariant homotopy, then the corresponding L^2-Betti numbers of $\widetilde{K}/\Gamma = K$ and $\widetilde{L}/\Gamma = L$ are equal.
- (Cohen, [3])
$$\sum_{p=0}^{n}(-1)^p L_{(2)}b^p(K) = e(K),$$
the Euler characteristic of K.
- (Cheeger and Gromov, [2]) If $\widetilde{K}$ is contractible and Γ is amenable, then all L^2-Betti numbers are vanishing.
- (Linnell, [11]) If Γ is elementary amenable and torsion-free, then all L^2-Betti numbers are integers.
- (Lũck, [12]) Let Γ be residually finite and
$$\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots, \quad \bigcap_{i=1}^{\infty} \Gamma_i = 1_\Gamma$$

normal subgroups of finite index and let $X_i = \widetilde{K}/\Gamma_i$ be the corresponding finite coverings of K. Then

$$L_{(2)}b^p(K) = \lim_{i\to\infty} \frac{\dim_{\mathbf{R}} H^p(X_i, \mathbf{R})}{|\Gamma : \Gamma_i|}.$$

- (Dodziuk and Mathai [5]) If $\{L_n\}_{n=1}^{\infty}$ is an exhaustion of $\widetilde{K}$ by finite simplicial complexes spanned by a $\{F_n\}_{n=1}^{\infty}$ Følner-exhaustion, then

$$L_{(2)}b^p(K) = \lim_{i\to\infty} \frac{\dim_{\mathbf{R}} H^p(L_n, \mathbf{R})}{|F_n|}.$$

Note that the second and the third results together imply that if K is an acyclic simplicial complex with amenable fundamental group, then its Euler characteristic is zero [2]. Another interesting application is due to Lück: If Γ is amenable, then the group algebra $\mathbf{C}[\Gamma]$ as a free module over itself generates an infinite cyclic subgroup in the Grothendieck group of $\mathbf{C}[\Gamma]$ [13].

The analogue setting we are investigating in this paper is the following. Let Γ be a finitely generated amenable group (see [6] why amenability is crucial). We denote by $\sum_\Gamma$ the full Bernoulli shift that is the linear space of $\mathbf{F}_2$-valued functions on Γ, where $\mathbf{F}_2$ is the field of two elements. The space $\sum_\Gamma$ is a compact, metrizable space in the pointwise convergence topology equipped with the natural left Γ-action by translations. A space $V \subset [\sum_\Gamma]^n$ is a linear subshift if it is linear as a $\mathbf{F}_2$-vector space, closed in the topology and invariant with respect to the Γ-action. The notion of dimension is the topological entropy of the linear subshifts. This is well-known for $\mathbf{Z}$ and $\mathbf{Z}^d$-actions and somehow less-known for general amenable group actions (nevertheless see [14]). We shall observe that our dimension h_Γ satisfies similar axioms as $\dim_\Gamma$:

1. **Nonnegativity**: For any V linear subshift : $h_\Gamma(V) \geq 0$. But it can be zero even if V is not zero.
2. **Monotonicity**: If $V \subset W$, then $h_\Gamma(V) \leq h_\Gamma(W)$.
3. **Invariance**: If $T : V \to W$ continuous Γ-equivariant linear isomorphism, then $h_\Gamma(V) = h_\Gamma(W)$.
4. **Additivity**: If $Z = V \oplus W$, then $h_\Gamma(Z) = h_\Gamma(V) + h_\Gamma(W)$.
5. **Continuity**: If $V_1 \supset V_2 \supset \cdots$ is a decreasing sequence of linear subshifts, then

$$h_\Gamma(\bigcap_{j=1}^{\infty} V_j) = \lim_{j\to\infty} h_\Gamma(V_j).$$

6. **Normalization**: $h_\Gamma(\sum_\Gamma) = 1$.

Now let $\widetilde{K}$ be as above. Then we have the ordinary cochain complex of $\mathbf{F}_2$-coefficients over $\widetilde{K}$:

$$C^0(\widetilde{K},\mathbf{F}_2) \xrightarrow{d_0} C^1(\widetilde{K},\mathbf{F}_2) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n(\widetilde{K},\mathbf{F}_2).$$

Then the p-cochain space $C^p(\widetilde{K},\mathbf{F}_2)$ is Γ-isomorphic to $[\sum_\Gamma]^{|K_p|}$, where K_p denotes the set of p-simplices in K. We define the p-th entropy Betti number $b_E^p(K)$ as $h_\Gamma(\operatorname{Ker} d_p) - h_\Gamma(\operatorname{Im} d_{p-1})$. In this paper we shall prove the following analogues of the L^2-results.

- If $\widetilde{K}$ and $\widetilde{L}$ are homotopic by a Γ-invariant homotopy and Γ is poly-cyclic, then the corresponding entropy-Betti numbers of $\widetilde{K}/\Gamma = K$ and $\widetilde{L}/\Gamma = L$ are equal.
- $$\sum_{p=0}^{n}(-1)^p b_E^p(K) = e(K),$$
  the Euler characteristic of K.
- If $\widetilde{K}$ is contractible, then all entropy-Betti numbers are vanishing. (This is quite obvious; the point is that the corollary on the vanishing Euler-characteristic still follows from this and the previous statement.)
- If Γ is poly-infinite-cyclic, then all entropy-Betti numbers are integers.
- Let Γ be free Abelian and
  $$\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots, \quad \bigcap_{i=1}^{\infty} \Gamma_i = 1_\Gamma$$
  normal subgroups of finite index and let $X_i = \widetilde{K}/\Gamma_i$ be the corresponding finite coverings of K. Then
  $$b_E^p(K) = \lim_{i\to\infty} \frac{\dim_{\mathbf{F}_2} H^p(X_i,\mathbf{F}_2)}{|\Gamma : \Gamma_i|}.$$
- If $\{L_n\}_{n=1}^{\infty}$ is an exhaustion of $\widetilde{K}$ by finite simplicial complexes spanned by a $\{F_n\}_{n=1}^{\infty}$ Følner-exhaustion, then
  $$b_E^p(K) = \lim_{n\to\infty} \frac{\dim_{\mathbf{F}_2} H^p(L_n,\mathbf{F}_2)}{|F_n|}.$$

We shall also prove an analogue of Lück's result on the Grothendieck-group for the group algebras $\mathbf{F}_2[\Gamma]$.

2. Amenable groups and quasi-tiles

Let Γ be a finitely generated group with a symmetric generator set $\{g_1, g_2, \ldots, g_k\}$. The right Cayley-graph of Γ, C_Γ, is defined as follows. Let $V(C_\Gamma) = \Gamma$, $E(C_\Gamma) = \{(a,b) \in \Gamma \times \Gamma\text{: there exists } g_i : ag_i = b\}$. The shortest path distance d of C_Γ makes Γ a discrete metric space. We shall use the following notation. If $H \subset \Gamma$ is a finite set, then $B_r(H)$ is the set of elements a in Γ such that there exists $h \in H, d(a,h) \leq r$. We denote $B_1(H)\backslash H$ by ∂H and $B_r(H)\backslash H$ by $\partial_r H$. An exhaustion of Γ by finite sets

$$1_\Gamma \in F_1 \subset F_2 \subset \cdots, \quad \bigcup_{j=1}^{\infty} F_j = \Gamma$$

is called a Følner-exhaustion if for any $r \in \mathbf{N}$: $\lim_{n\to\infty}(|\partial_r F_n|/|F_n|) = 0$. A group Γ is called amenable if it possesses a Følner-exhaustion. Some amenable groups have **tiling** Følner-exhaustion, that is any F_n is a tile: There exists $C \subset \Gamma$ such that $\{cF_n\}_{c\in C}$ is a partition of Γ. For example, $\mathbf{Z}^n$ has this tiling property. As observed by Ornstein and Weiss [15] **any** amenable group has quasi-tiling Følner-exhaustion. Let us recall their construction. Let $\{A_i\}_{i=1}^{\infty}$ be finite sets. Then we call them ϵ-disjoint if there exist subsets $\overline{A_i} \subset A_i$ so that $\overline{A_i} \cap \overline{A_j} = 0$ if $i \neq j$, and $|\overline{A_i}|/|A_i| \geq 1-\epsilon$ for all i. Now let B another finite set. We say that $\{A_i\}_{i=1}^{\infty}$ $(1-\epsilon)$-cover B, if

$$\frac{|B \cap \bigcup_{i=1}^{\infty} A_i|}{|B|} \geq 1 - \epsilon.$$

The subsets of Γ, $1_\Gamma \in T_1 \subset T_2 \subset \cdots \subset T_N$, form an ϵ-quasi-tile system if for any finite subset of $A \subset \Gamma$ there exists $C_i \subset \Gamma$, $i = 1, 2, \ldots, N$ such that

1. $C_iT_i \cap C_jT_j = \emptyset$ if $i \neq j$.
2. $\{cT_i : c \in C_i\}$ are ϵ-disjoint sets for any fixed i.
3. $\{C_iT_i, 1 \geq i \geq N\}$ form a $(1-\epsilon)$-cover of A.

The following proposition is Theorem 6 in [15]:

PROPOSITION 2.1: *If $F_1 \subset F_2 \subset \cdots$ is a Følner-exhaustion of an amenable group, then for any $\epsilon > 0$ we can choose a finite subset $F_{n_1} \subset F_{n_2} \subset \cdots \subset F_{n_N}$ such that they form an ϵ-quasi-tile system. The number N may depend on ϵ.*

3. The topological entropy of linear subshifts

First of all we define an averaged dimension $h_\Gamma(W)$ for linear subshifts and then we shall show that it coincides with the topological entropy. Let Γ be a finitely generated amenable group with Følner-exhaustion $1_\Gamma \in F_1 \subset F_2 \subset \cdots$, $\bigcup_{j=1}^\infty F_j = \Gamma$. We introduce some notations. If $\Lambda \subset \Gamma$ is a finite set, then let $[\sum_\Lambda]^r$ be the space of functions in $[\sum_\Gamma]^r$ supported on Λ. Also, $[\sum_\Gamma^0]^r$ denotes the space of finitely supported functions. Now let $W \subset [\sum_\Gamma]^r$ be a Γ-invariant not necessarily closed linear subspace. Then for any finite $\Lambda \subset \Gamma$, let $W_\Lambda \subset [\sum_\Lambda]^r$ be the linear space of functions η supported on Λ such that there exists $\nu \in W$: $\eta\,|_\Lambda = \nu\,|_\Lambda$.

Definition 3.1: $h_\Gamma(W) = \limsup_{n\to\infty}(\log_2 |W_{F_n}|/|F_n|)$

Note that $\log_2 |W_{F_n}|$ is just the dimension of the vector space W_{F_n} over the field $\mathbf{F}_2$. It will be obvious from the next proposition that $h_\Gamma(W)$ does not depend on the particular choice of the exhaustion.

PROPOSITION 3.1: 1. *$h_\Gamma(W) = h_\Gamma(\overline{W})$, where $\overline{W}$ denotes the closure of W in the pointwise convergence topology.*

2. *$h_\Gamma(W) = \liminf_{n\to\infty}(\log_2 |W_{F_n}|)/|F_n|$, hence $\lim_{n\to\infty}(\log_2 |W_{F_n}|)/|F_n|)$ always exists and equals $h_\Gamma(W)$.*

Proof: The first part is obvious from the definition. For the second part we argue by contradiction. Suppose that

$$h_\Gamma(W) - \liminf_{n\to\infty} \frac{\log_2 |W_{F_n}|}{|F_n|} = \delta > 0.$$

Consider a subsequence $F_{n_1} \subset F_{n_2} \subset \cdots$ such that

$$\sup_{i\to\infty} \frac{\log_2 |W_{F_{n_i}}|}{|F_{n_i}|} \le \liminf_{n\to\infty} \frac{\log_2 |W_{F_n}|}{|F_n|} + \epsilon,$$

where the explicit value of ϵ shall be chosen later accordingly. Then pick an ϵ-quasi-tile system from our subsequence: $F_{m_1} \subset F_{m_2} \subset \cdots \subset F_{m_N}$. Now we take an arbitrary F_n from the original Følner-exhaustion. By Proposition 2.1 we have an ϵ-disjoint $(1-\epsilon)$-covering of F_n by translates of the quasi-tile system. Denote by $R_1, R_2, \ldots, R_k$ those tiles which are properly contained in F_n. Then we have the following estimate,

$$|W_{F_n}| \le 2^{(r\epsilon|F_n| + r|F_n \backslash B_{D+1}(\partial F_n)|)} \prod_{i=1}^k |W_{R_i}|, \tag{1}$$

where D is the diameter of the largest tile F_{m_N}. The inequality (1) follows from the fact that a function $\xi \in W_{F_n}$ is uniquely determined by its restrictions on the covering tiles and its restriction on the uncovered elements. The latter one consists of two parts: the elements which are not covered at all by the original covering and the elements which are covered only by tiles intersecting the complement of F_n. These "badly" covered elements must be in a $(D+1)$-neighbourhood of the boundary of F_n. Also, by ϵ-disjointness we have the estimate

$$\sum_{i=1}^{k} |R_i| \leq \frac{1}{1-\epsilon}|F_n|. \tag{2}$$

Therefore,

$$|W_{F_n}| \leq 2^{(r\epsilon|F_n|+r|F_n\backslash B_{D+1}(\partial F_n)|)} 2^{\frac{1}{1-\epsilon}(h_\Gamma-\delta+\epsilon)|F_n|}.$$

Hence,

$$\frac{\log_2|W_{F_n}|}{|F_n|} \leq r\epsilon + \frac{r|F_n\backslash B_{D+1}(\partial F_n)|)}{|F_n|} + \frac{1}{1-\epsilon}(h_\Gamma - \delta + \epsilon). \tag{3}$$

Consequently, if we choose ϵ small enough, then for large n,

$$\frac{\log_2|W_{F_n}|}{|F_n|} \leq h_\Gamma - \frac{\delta}{2},$$

leading to a contradiction. ∎

Now we recall the notion of topological entropy. Let Γ be an amenable group as above and let X be a compact metric space equipped with a continuous Γ-action; $\alpha\colon \Gamma \to \mathrm{Homeo}(X)$. Instead of the original definition of Moulin Ollagnier [14] we use the equivalent "spanning-separating" definition, that is a direct generalization of the Abelian case [20]. We call a finite set $S \subset X$ (n,ϵ)-separating if for any distinct points $s,t \in S$ there exists $\gamma \in F_n$ such that $d(\alpha(\gamma^{-1})(s), \alpha(\gamma^{-1})(t)) > \epsilon$. We denote by $s(n,\epsilon)$ the maximal cardinality of such sets. We call a finite set $R \subset X$ (n,ϵ)-spanning if for any $x \in X$ there exists $y \in R$ such that $d(\alpha(\gamma^{-1})(x), \alpha(\gamma^{-1})(y)) \leq \epsilon$, for all $\gamma \in F_n$. We denote by $r(n,\epsilon)$ the minimal cardinality of such sets. Obviously, if $\epsilon' < \epsilon$ then $s(n,\epsilon') \geq s(n,\epsilon), r(n,\epsilon') \geq r(n,\epsilon)$. Also, we have the inequalities

$$r(n,\epsilon) \leq s(n,\epsilon) \leq r(n,\varepsilon/2).$$

Indeed, any maximal (n,ϵ)-separating set is (n,ϵ)-spanning. On the other hand if $R = x_1, x_2, \ldots, x_k$ is a minimal $(n,\varepsilon/2)$-spanning set, then

$$X = \bigcup_{i=1}^{k} D(x_i, n, \varepsilon/2),$$

where

$$D(x_i, n, \varepsilon/2) = \{y \in X: d(\alpha(\gamma^{-1})(x), \alpha(\gamma^{-1})(y)) \leq \varepsilon/2, \text{for all } \gamma \in F_n\}.$$

Any $D(x_i, n, \varepsilon/2)$ can contain at most one element of a (n, ϵ)-separating set, hence $s(n, \epsilon) \leq r(n, \varepsilon/2)$.

Consequently,

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log_2 r(n,\epsilon)}{|F_n|} = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log_2 s(n,\epsilon)}{|F_n|}.$$

This joint limit is called the topological entropy of the Γ-action and denoted by $h^{top}_\alpha(X)$.

Proposition 3.2: *Let $V \subset [\sum_\Gamma]^r$ be a linear subshift. Then $h^{top}_L(V) = h_\Gamma(V)$.*

Proof: First we fix a metric on V that defines the pointwise convergence topology. If $v, w \in V$, then let $d_V(v, w) = 2^{-(n-1)}$, where n is the infimum of k's such that $v\,|_{F_k} \neq w\,|_{F_k}$. First note that $|V_{F_n}| \leq s(n, 1)$. Indeed if $v_1, v_2, \ldots, v_s$ is a subset of V such that $v_i\,|_{F_n} \neq v_j\,|_{F_n}$ when $i \neq j$, then there exists $\gamma \in F_n$ such that $L_{\gamma^{-1}} v_i(1_\Gamma) \neq L_{\gamma^{-1}} v_j(1_\Gamma)$. Fix an ϵ and choose $K_\epsilon, M_\epsilon \in \mathbf{N}$ such that $\epsilon > 2^{-K_\epsilon}$ and $F_{K_\epsilon} \subset B_{M_\epsilon}(1_\Gamma)$. Then we claim that $s(n, \epsilon) \leq |V_{B_{M_\epsilon}(F_n)}|$. Indeed, if $x\,|_{B_{M_\epsilon}(F_n)} = y\,|_{B_{M_\epsilon}(F_n)}$, then for any $\gamma \in F_n$, $d_V(L_{\gamma^{-1}}(x), L_{\gamma^{-1}}(y)) \leq 2^{-K_\epsilon} < \epsilon$. Hence if $\epsilon < 1$,

$$h_\Gamma(V) = \lim_{n\to\infty} \frac{\log_2 |V_{F_n}|}{|F_n|} \leq \log_2(s(n,\epsilon)) \leq \lim_{n\to\infty} \frac{\log_2 |V_{B_{M_\epsilon}(F_n)}|}{|F_n|} = h_\Gamma(V). \quad \blacksquare$$

4. Extended configurations

The notion of extended configuration is due to Ruelle in a slightly different form. Again we start with a linear subshift $V \subset [\sum_\Gamma]^r$. For any $\Lambda \subset \Gamma$ finite set let $V^\Omega_\Lambda \subset [\sum_\Lambda]^r$ be a finite dimensional linear subspace satisfying the following axioms:

- **Extension**: $V_\Lambda \subset V^\Omega_\Lambda$.
- **Invariance**: $V^\Omega_{\gamma\Lambda} = L_\gamma(V^\Omega_\Lambda)$.
- **Transitivity**: If $\Lambda \subset M$, then for any $\xi \in V^\Omega_M$ there exists $\mu \in V^\Omega_\Lambda$ such that $\xi\,|_\Lambda = \mu\,|_\Lambda$.
- **Determination**: If $\xi \in [\sum_\gamma]^r$ and for any $\Lambda \subset \Gamma$ finite there exists $\xi_\Lambda \in V^\Omega_\Lambda$ such that $\xi\,|_\Lambda = \xi_\Lambda$, then $\xi \in V$.

We call such a system an extended configuration of V. Its topological entropy is defined as $h^\Omega_\Gamma(V) = \limsup_{n\to\infty} |V^\Omega_{F_n}|/|F_n|$.

PROPOSITION 4.1: *$h_\Gamma^\Omega(V) = h_\Gamma(V)$ (compare to Theorem 3.6 [19])*

Proof: Let us suppose that $h_\Gamma^\Omega(V) - h_\Gamma(V) = \delta > 0$. Again choose an ϵ-quasi-tile system $F_{n_1}, F_{n_2}, \ldots, F_{n_N}$ such that $|V_{F_{n_i}}| < 2^{(h_\Gamma+\epsilon)|F_{n_i}|}$, where the explicit value of ϵ will be given later. Denote by $V^k_{F_{n_i}}$ the space of functions μ in $V_{F_{n_i}}$ such that there exists $\xi \in V^\Omega_{B_k(F_{n_i})}$ with $\xi\,|_{F_{n_i}} = \mu\,|_{F_{n_i}}$. By Extension and Determination properties it is easy to see that for large p

$$V^p_{F_{n_i}} = V_{F_{n_i}}, \tag{4}$$

where $1 \leq i \leq N$. Let us pick a large p. Then we can proceed almost the same way as in the previous section. Denote by R_i, $1 \leq i \leq k$ those translates in the ϵ-disjoint $(1-\epsilon)$-covering of a Følner-set F_n such that not only the R_i's but even the $B_p(R_i)$ balls are contained in F_n. Then by Transitivity we have the following estimate,

$$|V^\Omega_{F_n}| \leq 2^{(r\epsilon|F_n|+r|F_n\backslash B_{p+D+1}(\partial F_n)|)} \prod_{i=1}^k |V^p_{R_i}|.$$

That is by Invariance, (2) and (4),

$$\frac{\log_2|V^\Omega_{F_n}|}{|F_n|} \leq r\epsilon + \frac{r|F_n\backslash B_{p+D+1}(\partial F_n)|}{|F_n|} + \frac{1}{1-\epsilon}(h^\Omega_\Gamma - \delta + \epsilon)$$

which leads to a contradiction provided that we choose ϵ small enough. ∎

5. Basic properties

Now we are in a position to prove the basic properties of h_Γ as stated in the Introduction. The Monotonicity, Normalization and Positivity axioms are obviously satisfied.

LEMMA 5.1: *Let $V, W \subset [\sum_\Gamma]^r$ be linear subshifts such that $V \cap W = 0$. Then $h_\Gamma(V) + h_\Gamma(W) = h_\Gamma(V \oplus W)$.*

Proof: First note that just because $V \cap W$ is the zero subspace it is not necessarily true that $V_\Lambda \cap W_\Lambda = 0$ as well. However, we can prove that $N^\Omega_\Lambda = V_\Lambda \cap W_\Lambda$ is an extended configuration of the zero subspace. We only need to show that the Determination axiom is satisfied. Suppose $\xi \in [\sum_\Gamma]^r$ such that $\xi\,|_{F_n} \in V_{F_n} \cap W_{F_n}$. Then there exists $v_n \in V, w_n \in W$ such that $\xi\,|_{F_n} = v_n\,|_{F_n} = w_n\,|_{F_n}$. Hence $v_n \to \xi, w_n \to \xi$ in the topology of $[\sum_\Gamma]^r$. The spaces V and W are closed, thus $\xi \in V \cap W$, hence $\xi = 0$. By elementary linear algebra,

$$\dim_{\mathbf{F}_2}(V \oplus W)_\Lambda + \dim_{\mathbf{F}_2} N^\Omega_\Lambda = \dim_{\mathbf{F}_2} V_\Lambda + \dim_{\mathbf{F}_2} W_\Lambda.$$

Hence by our Proposition 4.1 our Lemma follows. ∎

Now we prove a property of the entropy that is slightly more general than invariance.

PROPOSITION 5.1: *Let $T\colon V \to W$ be a continuous Γ-equivariant linear map between linear subshifts $V \subset [\sum_\Gamma]^r, W \subset [\sum_\Gamma]^r$. Then $h_\Gamma(\operatorname{Ker} T) + h_\Gamma(\operatorname{Im} T) = h_\Gamma(V)$.*

Proof: First of all let us note that by the compactness of V and the continuity of T both $\operatorname{Ker} T$ and $\operatorname{Im} T$ are linear subshifts. Now let us consider the natural right action of $\mathbf{F}_2(\Gamma)$ on $[\sum_\Gamma]^r$, $R_\gamma f(x) = f(x\gamma)$. This action obviously commutes with our previously defined left Γ-action. The right action can be extended to $s \times r$ matrices with coefficients in $\mathbf{F}_2(\Gamma)$ acting on the column vectors $[\sum_\Gamma]^r$. Obviously, any such matrix M defines a Γ-equivariant map; $T_M\colon [\sum_\Gamma]^r \to [\sum_\Gamma]^s$.

LEMMA 5.2: *Any continuous Γ-equivariant linear map $T\colon V \to W$ can be given via multiplication by some $s \times r$ matrix T_M with coefficients in $\mathbf{F}_2(\Gamma)$.*

Proof: Since T is uniformly continuous the value of $T(v)(1_\Gamma)$ is determined by the value of v on a finite ball B, where B does not depend on v. Hence for any $1 \leq i \leq s$

$$T(v)(1_\Gamma) = \sum_{\gamma \in B} \sum_{j=1}^{r} c_{ij}^{\gamma} \cdot v_j(\gamma),$$

where $c_{ij}^\gamma \in \mathbf{F}_2$. By $\{T_M\}_{ij} = \{\sum_{\gamma\in B} c_{ij}^\gamma \gamma\} \in \operatorname{Mat}_{s\times r}(\mathbf{F}_2[\Gamma])$ define a $s \times r$ matrix. Then for any $v \in V, T_M(v)(1_\Gamma) = T(v)(1_\Gamma)$. Hence by the Γ-equivariance of the matrix multiplication

$$\begin{aligned} T_M(v)(\gamma) &= L_{\gamma^{-1}}(T_M(v))(1_\Gamma) = T_M(L_{\gamma^{-1}}(v))(1_\Gamma) \\ &= T(L_{\gamma^{-1}}(v))(1_\Gamma) = T(v)(\gamma). \end{aligned}$$ ∎

Now we return to the proof of our Proposition. We denote by T_M the matrix and by k the diameter of the ball B defined in our Lemma. Let

$$N_\Lambda^\Omega = \{v \in [\textstyle\sum_\Lambda]^r : \text{ there exists } z \in V_{B_k(\Lambda)}, \text{ such that } z\,|_\Lambda = v\,|_\Lambda \text{ and } T(z)\,|_\Lambda = 0\}$$

and

$$M_\Lambda^\Omega = \{w \in [\textstyle\sum_\Lambda]^s : \text{ there exists } z \in V_{B_k(\Lambda)}, \text{ such that } T(z)\,|_\Lambda = w\,|_\Lambda\}.$$

Then N^Ω_Λ is an extended configuration of $\mathrm{Ker}\, T_M$, M^Ω_Λ is an extended configuration of $\mathrm{Im}\, T_M$. Let $\widetilde{T}_\Lambda\colon V_{B_k(\Lambda)} \to [\sum_\Lambda]^s$ be the restriction of T onto Λ. Then $\mathrm{Im}\, \widetilde{T}_\Lambda = M^\Omega_\Lambda$. We have the usual pigeon-hole estimate:

$$(5) \qquad |N^\Omega_{F_n}| \le |\,\mathrm{Ker}\, \widetilde{T}_{F_n}| \le |N^\Omega_{F_n}| 2^{r|B_k(\partial F_n)|}.$$

Also, by linear algebra we obtain

$$\dim_{\mathbf{F}_2} \mathrm{Ker}\, \widetilde{T}_{F_n} + \dim_{\mathbf{F}_2} \mathrm{Im}\, \widetilde{T}_{F_n} = \dim_{\mathbf{F}_2} V_{B_k(F_n)},$$

that is

$$(6) \qquad \log_2 |\,\mathrm{Ker}\, \widetilde{T}_{F_n}| + \log_2 |\,\mathrm{Im}\, \widetilde{T}_{F_n}| = \log_2 |V_{B_k(F_n)}|.$$

It is easy to see that (5) and (6) together imply the statement of our Proposition. ∎

Now we prove the Continuity property.

PROPOSITION 5.2: *If $V^1 \supset V^2 \supset \cdots$ is a decreasing sequence of linear subshifts, then*

$$h_\Gamma\Big(\bigcap_{j=1}^\infty V^j\Big) = \lim_{j\to\infty} h_\Gamma(V^j).$$

Proof: For any $k \in N, V^k \supset \bigcap_{j=1}^\infty V^j$. Hence

$$h_\Gamma\Big(\bigcap_{j=1}^\infty V^j\Big) \le \lim_{j\to\infty} h_\Gamma(V^j).$$

We need to prove the converse inequality. Suppose that for all j, $h_\Gamma(V^j) \ge h_\Gamma(\bigcap_{j=1}^\infty V^j) + 2\epsilon$. Let $T(j)$ be a monotone increasing function such that

$$\frac{\log_2 |V^j_{F_s}|}{|F_s|} \ge h_\Gamma\Big(\bigcap_{j=1}^\infty V^j\Big) + \epsilon,$$

when $s \ge T(j)$. We define a monotone non-decreasing function $S\colon \mathbf{N} \to \mathbf{N}$ such that $S(n) = 1$ if there is no such j so that $T(j) \le n$, and

$$S(n) = \max\{j : T(j) \le n\}$$

otherwise. Then $S(n) \to \infty$ as $n \to \infty$. Define $V^\Omega_\Lambda = V^{S(n)}_\Lambda$, where n is the smallest integer such that $|F_n| \ge |\Lambda|$. It is easy to see that V^Ω_Λ is an extended configuration of $\bigcap_{j=1}^\infty V^j$. Therefore, $(\log_2 |V^\Omega_{F_n}|)/|F_n| \to h_\Gamma(\bigcap_{j=1}^\infty V^j)$. On the other hand, by our construction $(\log_2 |V^\Omega_{F_n}|)/|F_n| \ge h_\Gamma(\bigcap_{j=1}^\infty V^j) + \epsilon$, leading to a contradiction. ∎

6. Pontryagin duality

In this section we recall the Pontryagin duality theory [8]. Let A be a locally compact Abelian group and let $\widehat{A}$ be its dual, that is the group of continuous homomorphisms $\chi\colon A \to S^1 = \{z \in C\colon |z| = 1\}$. According to the duality theorem A is naturally isomorphic to its double dual. The relevant example for us is $A = [\sum_\Gamma]^r$; its dual is $[\sum_\Gamma^0]^r$ the group of finitely supported elements, where $\langle \chi, f\rangle = \sum_{\gamma\in\Gamma}(\chi(\gamma), f(\gamma))$ for $\chi \in [\sum_\Gamma^0]^r$ and $f \in [\sum_\Gamma]^r$. Here $(\underline{a}, \underline{b})$ is defined as $\sum_{i=1}^r a_i b_i$. The additive group of $\mathbf{F}_2$ is viewed as the subgroup $\{-1, 1\} \in S^1$. If $H \subset [\sum_\Gamma]^r$ is a compact subgroup, then

$$H^\perp = \{\chi \in [\sum_\Gamma^0]^r : \langle \chi, h\rangle = 1, \text{for any } h \in H\}.$$

Conversely, if $B \subset [\sum_\Gamma^0]^r$ is a subgroup then

$$B^\perp = \{f \in [\sum_\Gamma]^r : \langle \chi, f\rangle = 1 \text{ for any } \chi \in B\}.$$

Then $(H^\perp)^\perp = H, (B^\perp)^\perp = B$. If A, B locally compact groups and $\psi\colon A \to B$ continuous homomorpisms, then its dual $\widehat{\psi}\colon \widehat{B} \to \widehat{A}$ is defined by $\langle \widehat{\psi}(\chi), a\rangle = \langle \chi, \psi(a)\rangle$. Again the double dual of ψ is itself if A and B are both compact or both discrete. Then ψ is injective resp. surjective if and only if $\widehat{\psi}$ is surjective (resp. injective). Moreover, if we have a short exact sequence of compact or discrete groups

$$1 \to A_1 \to A_2 \to \cdots \to A_n \to 1,$$

then its dual sequence

$$0 \to \widehat{A}_n \to \cdots \to \widehat{A}_2 \to \widehat{A}_1 \to 0$$

is also exact. The next proposition is a version of a result of Kitchens and Schmidt [9].

PROPOSITION 6.1: *The Pontryagin duality provides a one-to-one correspondence between linear subshifts and finitely generated left* $\mathbf{F}_2[\Gamma]$*-modules.*

Proof: First note that if L_γ is the left multiplication by γ on $[\sum_\Gamma]^r$, then $\widehat{L}_\gamma$ is the left multiplication by γ^{-1} on $[\sum_\Gamma^0]^r$. Hence if $V \xrightarrow{i} [\sum_\Gamma]^r$ is the natural imbedding of a linear subshift, then $(\mathbf{F}_2[\Gamma])^r \cong [\sum_\Gamma^0]^r \xrightarrow{i^*} \widehat{V}$ is a surjective $\mathbf{F}_2[\Gamma]$-module homomorpism, that is $\widehat{V}$ is a finitely generated left $\mathbf{F}_2[\Gamma]$-module. Conversely, the dual of a finitely generated $\mathbf{F}_2[\Gamma]$-module is a linear subshift. It is important to note that if $V \subset [\sum_\Gamma]^r$, $W \subset [\sum_\Gamma]^s$ are isomorphic linear subshifts, then the dual of this isomorphism provides a module-isomorphism between $\widehat{W}$ and $\widehat{V}$. Conversely, the duals of isomorphic modules are isomorphic linear subshifts. ∎

7. The Noether property of group algebras

Let $V \subset [\sum_\Gamma]^r$ be a linear subshift. We denote by V^0 the subspace of finitely supported elements.

Proposition 7.1: *If V^0 contains a non-zero element then $h_\Gamma(V^0) > 0$.*

Proof: Let us suppose that a ball $B_r(1_\Gamma)$ contains the support of a non-zero element in V^0. We claim that there exists an $\epsilon > 0$ such that if n is large enough, then F_n contains at least $\epsilon|F_n|$ disjoint translates of $B_r(1_\Gamma)$. First note that the claim implies our Proposition. If we have M_n translates of $B_r(1_\Gamma)$ in F_n, then we can find 2^{M_n} different elements of V^0 which are all supported in F_n. Therefore

$$\frac{\log_2 |V_{F_n}|}{|F_n|} \geq \frac{M_n}{|F_n|} \geq \frac{\epsilon|F_n|}{|F_n|} = \epsilon.$$

Hence $h_\Gamma(V^0) \geq \epsilon$. Let us prove the claim. Pick a maximal $2r$-net $a_1, a_2, \dots, a_{M_n}$, that is a maximal set of points in F_n such that any two have distance greater than or equal to $2r$. Then the $4r$-balls around the points a_i are covering F_n. Hence $M_n \geq |F_n|/B_{4r}(1_\Gamma)$. Then at least half of the a_i's are in $F_n \backslash B_{r+1}(\partial F_n)$. The balls around these elements being far from the boundary are completely in F_n. Hence we have at least $\frac{1}{2}|F_n|/B_{4r}(1_\Gamma)$ disjoint translates of $B_r(1_\Gamma)$ in F_n. ∎

In the rest of this section we shall have an extra assumption on the amenable group Γ. We call an amenable group Noether if $\mathbf{F}_2[\Gamma]$ is a Noether ring. That is, any left submodule of $(\mathbf{F}_2[\Gamma])^r$ is finitely generated. According to Hall's theorem [17], if Γ is polycyclic-by-finite then Γ is Noether.

Proposition 7.2: *If $V \subset [\sum_\Gamma]^r$ is a linear subshift and Γ is Noether, then $h_\Gamma(V) + h_\Gamma(V^\perp) = r$.*

First of all $V^\perp \subset [\sum_\Gamma^0]^r \subset [\sum_\Gamma]^r$, hence the expression $h_\Gamma(V^\perp)$ is meaningful. By our assumption $V^\perp$ is a finitely generated module, so let us choose a $\{r_1, r_2, \dots, r_k\}$ finite generator set. We need to prove that

$$\lim_{n\to\infty} \frac{\log_2 |V^\perp_{F_n}|}{|F_n|} = r - h_\Gamma(V).$$

In order to do so it is enough to see that

$$\lim_{n\to\infty} \frac{\log_2 |V^{\perp}_{F_n}| - \log_2 |V^{\perp}_n|}{|F_n|} = 0, \tag{7}$$

where $V^{\perp}_n$ denotes the set of elements in $V^{\perp}$, supported in F_n. Remember that $V^{\perp}_{F_n}$ denotes the restrictions of the elements of $V^{\perp}$, therefore $V^{\perp}_{F_n} \supset V^{\perp}_n$. By linear algebra,

$$\dim_{\mathbf{F}_2}(V_{F_n}) + \dim_{\mathbf{F}_2}(V^{\perp}_n) = r|F_n|,$$

that is

$$\lim_{n\to\infty} \frac{\log_2 |V^{\perp}_n|}{|F_n|} = r - h_\Gamma(V).$$

Let us prove (7). Any element of $V^{\perp}$ can be written (not in a unique way !) in the form $\sum_{i=1}^k a_i r_i$, where $a_i \in \mathbf{F}_2[\Gamma]$. Denote by D the supremum of the diameters of the r_i's. If $\operatorname{supp}(a_i) \subset F_n \backslash B_{D+1}(\partial F_n)$, for all i, then $\sum_{i=1}^k a_i r_i \in V^{\perp}_n$. On the other hand, if $\operatorname{supp}(a_i) \cap B_{D+1}(F_n) = 0$ for all i, then $\sum_{i=1}^k a_i r_i \,|_{F_n} = 0$. Therefore we have the pigeon-hole estimate

$$|V^{\perp}_{F_n}| \leq |V^{\perp}_n| 2^{krB_{D+1}(\partial F_n)},$$

that is

$$\frac{\log_2 |V^{\perp}_{F_n}| - \log_2 |V^{\perp}_n|}{|F_n|} \leq \frac{krB_{D+1}(\partial F_n)}{|F_n|},$$

and the right hand side tends to zero. ∎

Now we prove the density property.

Proposition 7.3: *If Γ is Noether and $V \subset [\sum_\Gamma]^r$ is a linear subshift, then $h_\Gamma(V) = h_\Gamma(V^0)$.*

Proof: By our previous Proposition, $h_\Gamma(V^{\perp}) = r - h_\Gamma(V)$. Therefore $h_\Gamma(\overline{V^{\perp}}) = r - h_\Gamma(V)$, where $\overline{V^{\perp}}$ is the closure of $V^{\perp}$ as $[\sum_\Gamma^0]^r$ imbeds into $[\sum_\Gamma]^r$. Using our previous Proposition again,

$$h_\Gamma((\overline{V^{\perp}})^{\perp}) = h_\Gamma(V).$$

If $\xi \in (\overline{V^{\perp}})^{\perp}$ then ξ is finitely supported and $\xi \in (V^{\perp})^{\perp} = V$, that is $\xi \in V^0$. Therefore,

$$h_\Gamma(V) = h_\Gamma((\overline{V^{\perp}})^{\perp}) \leq h_\Gamma(V^0) \leq h_\Gamma(V). \quad ∎$$

Actually, our proofs of the last two Propositions gives a slightly stronger result:

PROPOSITION 7.4: *Let Γ be Noether and let V be a linear subshift such that V^0 is generated by $r_1, r_2, \ldots, r_k$ as left $\mathbf{F}_2[\Gamma]$-module. Denote by $\widetilde{V}_n$ the set of those elements in V^0 which can be written in the form $\sum_{i=1}^k a_i r_i$, where all the a_i's are supported in F_n. Then $\lim_{n\to\infty}(\log_2|\widetilde{V}_n|)/|F_n| = h_\Gamma(V)$.*

8. The Yuzvinskii formula

Recall Yuzvinskii's additivity formula for Abelian groups [KS]. Let $\Gamma \cong \mathbf{Z}^d$ and α be a Γ-action of continuous automorphisms on a compact metric group X. Suppose that Y is a compact α-invariant subgroup; then

$$h_\alpha^{top}(X) = h_\alpha^{top}(Y) + h_\alpha^{top}(X/Y). \tag{8}$$

The results of Ward and Zhang [WZ] suggest that a similar statement might be true for general amenable actions. In our paper we prove only a very special case.

PROPOSITION 8.1: *Let $Y \subset X \subset [\sum_\Gamma]^r$ be linear subshifts where Γ is Noether. Then*

$$h_L^{top}(Y) + h_L^{top}(X/Y) = h_L^{top}(X) = h_\Gamma(X),$$

where L is the usual left Γ-action.

Proof: The key observation is the following lemma.

LEMMA 8.1: *Let $V \subset [\sum_\Gamma]^r$ be a linear subshift where Γ is Noether. Then there exists a constant D such that if $\xi \in [\sum_\Gamma]^r$ and $\xi\mid_{B_D(\gamma)} \in V_{B_D(\gamma)}$ for all $\gamma \in \Gamma$, then $\xi \in V$. That is, linear subshitfs are of finite type for Noether groups.*

Proof: Let $V^\perp \subset [\sum_\Gamma^0]^r$ be the orthogonal ideal of V. It is generated by $r_1, r_2, \ldots, r_N$, where all r_i's are supported in $B_D(1_\Gamma)$. Then $\xi \notin V$ if and only if $\langle \xi, L_\gamma(r_i)\rangle \neq 0$ for some i and $\gamma \in \Gamma$. It means that $\xi\mid_{B_D(\gamma)} \notin V_{B_D(\gamma)}$. ∎

Now we define a metric on X/Y. If $v, w \in X$ let $d([v],[w]) = 2^{-(n-1)}$, where n is the smallest integer such that $(v-w)\mid_{B_D(\gamma)} \notin Y_{B_D(\gamma)}$ for some $\gamma \in F_n$. Here D denotes the diameter of the joint support of a generator system $\{s_1, s_2, \ldots, s_M\}$ of the ideal $Y^\perp$.

LEMMA 8.2: *The metric d defines the pointwise convergence topology.*

Proof: We need to prove that $d([v_n], 0) \to 0$ implies that $[v_n] \to Y$ in the factor topology of X/Y (the converse is obvious). Suppose that $\{[v_n]\}$ does not converge to Y in the factor topology. Then there exists a subsequence v_{n_k} such that $v_{n_k} \to v \notin Y$ in the convergence topology of $[\sum_\Gamma]^r$. But then there exists a ball $B_D(\gamma)$ such that $v_{n_k}\mid_{B_D(\gamma)} \notin Y_{B_D(\gamma)}$ for large k. This contradicts the assumption that $d([v_{n_k}],0) \to 0$. ∎

Now let us turn back to the proof of our Proposition. Similarly to Proposition 3.2 we have a lower estimate for $s_{X/Y}(n,1)$ in the d-metric. Let us denote by G_n the set of elements in $Y^\perp$ which can be written in the form $\sum_{i=1}^m c_i s_i$ such that all the c_i's are supported in F_n. Denote by H_n the set of those elements of $[\sum_\Gamma]^r$ which are supported on $B_D(F_n)$ and orthogonal to G_n. Then by Propositions 7.3 and 7.4, $(\log_2 |H_n|)/|F_n| \to h_\Gamma(Y)$. We have the following inequality:

$$k_n = |X_{B_D(F_n)}|/|H_n \cap X_{B_D(F_n)}| \leq s_{X/Y}(n,1).$$

Indeed, there exist k_n elements of $X_{B_D(F_n)}$ such that their pairwise differences $x_i - x_j \notin H_n$, thus $\langle x_i - x_j, L_\gamma s_k\rangle \neq 0$ for some s_k and $\gamma \in F_n$. Hence $d(L_{\gamma^{-1}}([x_i]), L_{\gamma^{-1}}([x_j])) = 1$. Consequently, $h_\Gamma(X) - h_\Gamma(Y) \leq h_L^{top}(X/Y)$. Now fix an ϵ and let $B_R(1_\Gamma) \supset F_m$, where $2^{-m} < \epsilon$. Then obviously

$$s_{X/Y}(n,\epsilon) \leq \frac{|X_{B_{D+r}(F_n)}|}{|Y_{B_{D+r}(F_n)}|},$$

which implies the converse inequality $h_\Gamma(X) - h_\Gamma(Y) \geq h_L^{top}(X/Y)$.

9. Betti numbers

In this section we define an analogue of the L^2-Betti numbers. Let $\widetilde{K}$ be a regular, normal Γ-covering of a finite simplicial complex K, where Γ is an amenable group that acts freely and simplicially on $\widetilde{K}$ and $\widetilde{K}/\Gamma = K$. We have the ordinary cochain complex of $\mathbf{F}_2$-coefficients over $\widetilde{K}$:

$$C^0(\widetilde{K},\mathbf{F}_2) \xrightarrow{d_0} C^1(\widetilde{K},\mathbf{F}_2) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n(\widetilde{K},\mathbf{F}_2).$$

Then the p-cochain space $C^p(\widetilde{K},\mathbf{F}_2)$ is Γ-isomorphic to $[\sum_\Gamma]^{|K_p|}$, where K_p denotes the set of p-simplices in K. We define the p-th entropy Betti number $b_E^p(K)$ as $h_\Gamma(\operatorname{Ker} d_p) - h_\Gamma(\operatorname{Im} d_{p-1})$. The following theorem is the analogue of Cohen's theorem [3].

PROPOSITION 9.1: *$\sum_{p=0}^n (-1)^p b_E^p(K)$ equals the Euler-characteristics of K.*

Proof: By Proposition 5.1,

$$b_E^p(K) = h_\Gamma(\operatorname{Ker} d_p) + h_\Gamma(\operatorname{Ker} d_{p-1}) - h_\Gamma(C^{p-1}(\widetilde{K},\mathbf{F}_2)).$$

Summing up these equations for all p with alternating signs we obtain

$$\sum_{p=0}^{n}(-1)^p b_E^p(K) = \sum_{p=0}^{n}(-1)^p|K_p| = e(K). \quad \blacksquare$$

Now let us see the new proof of the result of Cheeger and Gromov.

PROPOSITION 9.2: *If $\widetilde{K}$ is contractible, then all entropy Betti numbers are vanishing.*

The proof is much easier than for the L^2-Betti numbers. If $p > 0$, then $\mathbf{F}_2$-cohomologies are vanishing, therefore $b_E^p(K) = 0$ if $p > 0$. If $p = 0$, then the cocycle space is finite so the entropy Betti number must be zero. $\blacksquare$

COROLLARY 9.1: *If K is a finite acyclic simplicial complex with an amenable fundamental group, then its Euler characteristic is zero.*

Now we prove the analogue of the result of Dodziuk and Mathai.

PROPOSITION 9.3:

$$b_E^p(K) = \lim_{n\to\infty} \frac{\dim_{\mathbf{F}_2} H^p(L_n, \mathbf{F}_2)}{|F_n|},$$

where $\{L_n\}$ is an exhaustion of $\widetilde{K}$ spanned by the Følner-sets.

Remark: Since $\dim_{\mathbf{F}_2} H^p(L_n, \mathbf{F}_2) \geq \dim_{\mathbf{R}} H^p(L_n, \mathbf{R})$, the entropy Betti numbers are at least as large as the corresponding L^2-Betti numbers. It is easy to construct examples, where some entropy Betti numbers are strictly larger than the corresponding L^2-Betti number (cf. the remark after Proposition 10.1).

Proof of Proposition 9.3: First note again that $C^p(\widetilde{K}, \mathbf{F}_2) \cong [\sum_\Gamma]^{|K_p|}$. Denote by R a constant such that for any $[(\gamma, p), (\delta, q)]$ 1-simplex of $\widetilde{K}$, $d_\Gamma(\gamma, \delta) \leq R$. Now we can build an extended configuration for $\operatorname{Ker} d_p$ and $\operatorname{Im} d_p$ in the following way. Let $S(\Lambda)$ be the simplicial complex spanned by vertices of the form (γ, p), where $\gamma \in B_{2R}(\Lambda)$ and $p \in K_0$. Also, let L_n be the simplicial complex spanned by the vertices with first coordinate in F_n. Consider the coboundary operator as $[\sum_\Gamma]^{|K_p|} \xrightarrow{d_p} [\sum_\Gamma]^{|K_{p+1}|}$. Let $A_p(\Lambda)$ be the space of those functions in $[\sum_\Gamma]^{|K_p|}$ which are supported on Λ and are the restriction of a cocycle of $S(\Lambda)$; respectively let $B_p(\Lambda)$ be the space of restrictions of coboundaries of $S(\Lambda)$. Obviously, $A_p(\Lambda)$ is an extended configuration of $\operatorname{Ker} d_p$ and $B_p(\Lambda)$ is an extended configuration of $\operatorname{Im} d_p$. Then the usual pigeon-hole argument and Proposition 4.1 imply that

$$h_\Gamma(\operatorname{Ker} d_p) = \lim_{n\to\infty} \frac{\dim_{\mathbf{F}_2}(Z^p(S(F_n)))}{|F_n|}, \quad h_\Gamma(\operatorname{Im} d_p) = \lim_{n\to\infty} \frac{\dim_{\mathbf{F}_2}(B^p(S(F_n)))}{|F_n|},$$

where Z^p resp. B^p denote the space of cocycles resp. coboundaries. Therefore

$$b_E^p(K) = \lim_{n\to\infty} \frac{\dim_{\mathbf{F}_2}(H^p(S(F_n),\mathbf{F}_2))}{|F_n|}.$$

Finally, we must prove that

$$\lim_{n\to\infty} \frac{\dim_{\mathbf{F}_2}(H^p(S(F_n),\mathbf{F}_2)) - \dim_{\mathbf{F}_2}(H^p(L_n,\mathbf{F}_2))}{|F_n|} = 0.$$

Note that it follows from the long exact cohomology sequence induced by the inclusion $L_n \to S(F_n)$ and the obvious fact that $(\dim_{\mathbf{F}_2}(H^p(S(F_n),L_n,\mathbf{F}_2))/|F_n|$ tends to zero as $n \to \infty$. ∎

10. Towers and fixed points

In this section we recall some ideas of Farber [7]. Let Γ be a finitely generated residually-2 group. That is, there exists a chain of normal subgroups of prime power index, $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$, where $\bigcap_{j=1}^{\infty} \Gamma_j = \{1_\Gamma\}$. Let $\widetilde{K}/\Gamma = K$ be as in the previous section. Then one can consider the tower of finite simplicial complexes $X_i = \widetilde{K}/\Gamma_i$. Note that X_i is a simplicial $(\Gamma:\Gamma_i)$-covering of K. Farber proved ([7], Theorem 11.1) that $\lim_{j\to\infty}(\dim_{\mathbf{F}_2} H^i(X_j,\mathbf{F}_2))/|\Gamma:\Gamma_i|$ always exists. The following conjecture is the analogue of Lück's theorem on approximating the L^2-Betti numbers [12]:

Conjecture 10.1: *If Γ is as above a residually-2 group, then*

$$\lim_{j\to\infty} \frac{\dim_{\mathbf{F}_2} H^i(X_j,\mathbf{F}_2)}{|\Gamma:\Gamma_i|} = b_E^i(K).$$

Proposition 10.1: *The conjecture is true if Γ is free Abelian.*

Proof: First of all note that if Γ is Noether, then any $V \subset [\sum_\Gamma]^r$ linear subshift is expansive. That is, there exists $\epsilon > 0$ such that if $x \neq y \in V$, then for some $\gamma \in \Gamma$, $d(L_\gamma(x), L_\gamma(y)) \geq \epsilon$. This is just a reformulation of Lemma 8.1. The following result is due to Lind, Schmidt and Ward [10].

Proposition 10.2: *If α is an expansive $\mathbf{Z}^d$-action by automorphisms of a compact Abelian group X, then*

$$\lim_{|\mathbf{Z}^d:\Lambda|\to\infty, \Lambda\subset\mathbf{Z}^d} \frac{|\operatorname{Fix}\Lambda|}{|\mathbf{Z}^d:\Lambda|} = h_\alpha^{top}(X)$$

where $\mathrm{Fix}\,\Lambda$ *denotes the set of fixed points of the subgroup* Λ.

Now let us turn to the proof of Proposition 10.1. Let Z_j^i be the space of i-cocycles on X_j and Z^i be the space of i-cocycles on $\widetilde{K}$. Then Z_j^i is exactly the set of fixed points of the subgroup Γ_j on Z^i. (Note that the similar statement on coboundaries would not be necessarily true.) By Proposition 10.2, $\lim_{j\to\infty}(\dim_{\mathbf{F}_2} Z_j^i)/|\Gamma:\Gamma_j| = h_\Gamma(Z^i)$. If C_j^i denotes the space of i-cochains on X_j and C^i denotes the space of i-cochains on $\widetilde{K}$, then $(\dim_{\mathbf{F}_2} C_j^i)/|\Gamma:\Gamma_j| = |K_i| = h_\Gamma(C^i)$, for all j. By our Proposition 5.1,

$$b_E^i(K) = h_\Gamma(Z^i) + h_\Gamma(Z^{i-1}) - h_\Gamma(C^{i-1}).$$

Also,

$$\dim_{\mathbf{F}_2} H^i(X_j,\mathbf{F}_2) = \dim_{\mathbf{F}_2}(Z_j^i) + \dim_{\mathbf{F}_2}(Z_j^{i-1}) - \dim_{\mathbf{F}_2}(C_j^{i-1}).$$

Hence our Proposition follows. ∎

It is not hard to construct a $\widetilde{K}$, where for some p the entropy and L^2-Betti numbers differ. Simply consider the Cayley graph of $\mathbf{Z}^d$ and then just stick a $\mathbf{R}P^4$ on each vertex. Then if L_n denote the approximative complexes for some Følner-exhaustion,

$$b_E^4(K) = \lim_{n\to\infty} \dim_{\mathbf{F}_2} \frac{H^4(L_n,\mathbf{F}_2)}{|F_n|} = 1$$

and

$$L_{(2)}b^4(K) = \lim_{n\to\infty} \dim_{\mathbf{R}} \frac{H^4(L_n,\mathbf{R})}{|F_n|} = 0.$$

11. The Grothendieck group and the integrality of the Betti numbers

First we recall the notion of the Grothendieck group of a non-commutative ring R [18]. Let $G(R)$ be the Abelian group, defined by generators $\{[M]\}$, where the M's are the finitely generated left R-modules up to isomorphism. The relations are in the form $[M]+[N]=[L]$, for any exact sequence $0\to M\to L\to N\to 0$. Lück [13] proved that if $R=\mathbf{C}[\Gamma]$, where Γ is amenable, then $[\mathbf{C}[\Gamma]]$ generates an infinite cyclic subgroup in $G(R)$.

PROPOSITION 11.1: $[\mathbf{F}_2[\Gamma]]$ *generates an infinite cyclic subgroup in* $G(\mathbf{F}_2[\Gamma])$ *for any finitely generated amenable group* Γ.

Proof: It is enough to define a rank on finitely generated $\mathbf{F}_2[\Gamma]$-modules, such that $\mathrm{rk}([\mathbf{F}_2[\Gamma]]) = 1$ and $\mathrm{rk}([M]) + \mathrm{rk}([N]) = \mathrm{rk}([L])$ if

$$0\to M \xrightarrow{i} L \xrightarrow{p} N \to 0.$$

Let $\mathrm{rk}(M) = h_\Gamma(\widehat{M})$. Now apply Proposition 5.1 for the subshifts

$$0 \to \widehat{N} \xrightarrow{\widehat{p}} \widehat{L} \xrightarrow{\widehat{i}} \widehat{M} \to 0$$

and the additivity follows. ∎

Linnell [11] proved that all L^2-Betti numbers are integers for torsion-free elementary amenable group Γ. We can prove the following proposition.

PROPOSITION 11.2: *If Γ is poly-infinite-cyclic, then $h_\Gamma(V)$ is an integer for any linear subshift V.*

Proof: Let $M = \widehat{V}$ be the dual $F_2[\Gamma]$-module of our subshift. Then, by Theorem 3.13 [17], M has a finite resolution by finitely generated projective modules:

$$0 \to M_n \to \cdots \to M_2 \to M_1 \to M \to 0.$$

Then, as we pointed out earlier, the dual sequence

$$0 \to V \to V_1 \to V_2 \to \cdots \to V_n \to 0$$

is an exact sequence of linear subshifts and continuous homomorphisms, where $V_i = \widehat{M_i}$. By Proposition 5.1 it is enough to show that all the $h_\Gamma(V_i)$'s are integers. By a result of Grothendieck and Serre (Theorem 4.13 [17]), if Γ is poly-infinite-cyclic, then all finitely generated, projective $\mathbf{F}_2[\Gamma]$-modules are stably free. Hence, using the notation of the previous section,

$$h_\Gamma(V_i) = \mathrm{rk}(\widehat{V_i}) = \mathrm{rk}((\mathbf{F}_2[\Gamma])^n) - \mathrm{rk}((\mathbf{F}_2[\Gamma])^m) = n - m$$

is an integer. ∎

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